

The Differential and Functional Equations for a Lie Group Homomorphism are Equivalent

George Svetlichny

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Abstract

I prove the “folklore” result that the functional equation for a Lie group homomorphism can be solved by solving the corresponding differential equation.

1 Introduction

The simplest example of our result concerns the functional equation:

$$f(x + y) = f(x)f(y)$$

As is well known, the measurable solutions with $f(0) = 1$ are exponentials: $f(x) = e^{kx}$. If we differentiate the equation in y and set $y = 0$ we get:

$$f'(x) = f'(0)f(x)$$

The solutions of this *differential* equation are the same exponential functions with $k = f'(0)$. A more complicated example occurred [1] when the I was faced with the problem of finding all lower triangular 2×2 matrices $M(X)$ depending on a vector X satisfying $M(0) = I$ and the functional equation:

$$M(X + M(X)Y) = M(X)M(Y)$$

The (differentiable) solutions were found to coincide with the solutions of the system of differential equations resulting from differentiating with respect to Y and setting $Y = 0$. Now it is far from common that a functional equation imply a differential equation, and even less common that all solutions of the differential equation be solutions of the original functional equation. What makes the trick work in the above cases is the fact that both equations can be construed as stating a Lie group homomorphism, and for these, as we shall shortly show, the functional and differential equations are equivalent, at least in a neighborhood of the identity, provided we are interested in differentiable solutions.¹ I have seen this result used on many occasions but have never seen a proof and so decided to provide one myself.

¹ For the functional equation it's probably enough to just require measurability.

2 The Result

All manifolds and maps are considered to be \mathcal{C}^∞ . Let G be a Lie group, U a neighborhood of the identity $e \in G$, and M a manifold of dimension m . Let $p : U \rightarrow M$ be a fibration and let $x_0 = p(e)$. Consider now the problem of finding in a neighborhood of x_0 a section σ of p such that:

$$\sigma(x_0) = e \quad (1)$$

$$\sigma(p(\sigma(x)\sigma(y))) = \sigma(x)\sigma(y) \quad (2)$$

Our two examples are instances of this. For the first example let \mathbb{R}^* be the multiplicative group of positive reals and take G to be the direct product $\mathbb{R}^* \times \mathbb{R}$ with p the projection onto the second factor; f is then the first component of σ . For the second example let T be the group of invertible lower triangular 2×2 matrices and take G to be $T \times \mathbb{R}^2$ with the product law:

$$(M, X) \cdot (N, Y) = (MN, X + MY),$$

and p be the projection on the second factor; as before the desired function is the first component of σ .

Now what (1-2) effectively say is that a neighborhood of x_0 is to acquire a local Lie group structure with the product given by:

$$x, y \mapsto p(\sigma(x)\sigma(y))$$

and that σ is to provide a homomorphism of this structure into G . Thus the image of σ is a local Lie subgroup of G of dimension m and transversal to the fiber $p^{-1}(x_0)$. Reciprocally, any local Lie subgroup with these two properties is a solution by taking σ to be the inverse of p restricted to this subgroup. The problem can be further reduced to an algebraic one. The tangent space \mathfrak{g} to G at e (the Lie algebra of G) has a natural vertical subspace (not necessarily a subalgebra) provide by the tangent space to the fiber. The germs of m -dimensional local Lie subgroups transversal to the fiber are now in one to one correspondence to the m dimensional Lie subalgebras of \mathfrak{g} that, as subspaces, are transversal to the vertical subspace. This in principle resolves the existence and uniqueness problem for germs at x_0 of solutions to (1-2). One hasn't though resolved the *practical* problem of finding such solutions explicitly, for even if the algebraic problem is solved one has to exponentiate the Lie subalgebra to find the map and this may not be a trivial task. Fortunately, the algebraic and exponentiation problems can be avoided by appealing to the differential equation that results from (2). For ease of notation let μ denote the group product of G . If we differentiate (2) with respect to y and set $y = x_0$ we obtain the following differential equation for σ :

$$\mathbb{D}\sigma(x) \cdot \mathbb{D}p(\sigma(x)) \cdot \mathbb{D}_2\mu(\sigma(x), e) \cdot \mathbb{D}\sigma(x_0) = \mathbb{D}_2\mu(\sigma(x), e) \cdot \mathbb{D}\sigma(x_0) \quad (3)$$

where \mathbb{D} denotes the Frechet derivative. Now obviously any solution to (1-2) necessarily satisfies (3), what is remarkable is that the converse is true.

Theorem 1 *Any local solution of (3) such that $\sigma(x_0) = e$ is a (local) solution of (1-2).*

Proof: By our previous discussion it's enough to show that the image S of σ is a local Lie subgroup of G . Apply both sides of (3) to a tangent vector η at x_0 . The left hand side is of the form $\mathbb{D}\sigma(x) \cdot \eta$ and is thus a tangent vector to S at $\sigma(x)$. The right-hand side is $\mathbb{D}_2\mu(\sigma(x), e) \cdot \mathbb{D}\sigma(x_0) \cdot \xi$. Now $\mathbb{D}\sigma(x_0) \cdot \xi$ is a tangent vector to S at e , and by picking ξ appropriately any such tangent vector can be so given. On the other hand $\mathbb{D}_2\mu(\sigma(x), e)$ is the tangent map of left multiplication by $\sigma(x)$. Thus a consequence of (3) is that the left translate by $\sigma(x)$ of a tangent vector to S at e is tangent to S at $\sigma(x)$, in other words: those left-invariant vector fields that are tangent to S at e are tangent to S (at all other points). Because of this tangency to the same submanifold, these vector fields are in involution and so form a Lie subalgebra. Associated to this subalgebra is a unique germ of a local Lie subgroup. Since any such subgroup has a neighborhood of the identity covered by exponentiations of the tangent left-invariant vector fields, we see that the subgroup germ coincides with the germ of S at e . Q.E.D

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References

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